Quantum state swapping via qubit network with Hubbard interaction

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We study the quantum state transfer (QST) in a class of qubit network with on-site interaction, which is described by the generalized Hubbard model with engineered couplings. It is proved that the system of two electrons with opposite spins in this quantum network of N sites can be rigorously reduced into N one dimensional engineered single Bloch electron models with central potential barrier. With this observation we find that such system can perform a perfect QST, the quantum swapping between two distant electrons with opposite spins. Numerical results show such QST and the resonant-tunnelling for the optimal on-site interaction strengths.

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Introduction. For implementing quantum information processing based on the scalable systems, the solid-state data bus is a necessary element to coherently integrate two or more qubits and transfer the quantum information among them [1]. Recently increasing investigations have explored the possibilities to transfer quantum states through a class of solid-state data buses, the artificial spin chain with engineered nearest neighbor (NN) couplings [2, 3, 4, 5, 6, 7, 8, 9]. Some novel physical mechanisms have been discovered behind the protocols of quantum state transfer (QST) based on the spin chain systems. For example, it is discovered that the gap structure of spectrum of the strongly correlated systems is responsible for the role of data bus [7]; and the spectrum marching parity symmetry is a sufficient condition for perfect QST [6, 9].

We also notice that most of the explorations for QST are carried out only for the non-interacting systems or single-particle quantum states. And it seems that the on-site Coulomb interactions may destroy the quantum coherence of transferred state. In this letter, we will study the influences of on-site interactions on the dynamic process of QST by making use of the generalized Hubbard model with engineered NN couplings as same as that in the artificial Bloch electron model in ref. [4].

To give prominence to our central context we only consider the simplest interacting system with only two electrons of opposite spin involved. The main result we achieved is the discovery of the novel model reduction that the N-site two-electron engineered Hubbard model can be decomposed into N single-particle engineered models on l-sites chain ($l = 1, 3, ..., 2N-1$), but with an additional central potential barrier (CPB). This discovery enlightens us to conjecture the possibility of implementing the perfect quantum information swapping since the reduced models still keep the mirror symmetry. The detailed numerical simulations demonstrate there indeed exists such perfect quantum state swapping even certain on-site repulsion U is considered.

Engineered Hubbard model and its reduction. Our model for quantum state swapping is the generalizations of the engineered spin model in ref. [4] (see the Fig. 1a) by adding the on-site Coulomb interactions. It can also be regarded as an engineered Hubbard model [10] with artificial hopping. The model Hamiltonian reads

$$H = -\sum_{j,\sigma}(t_j c^{\dagger}_{j,\sigma} c_{j+1,\sigma} + h.c.) + U\sum_{j} n_{j\uparrow} n_{j\downarrow}, \qquad (1)$$

where $c^{\dagger}_{j,\sigma}$ is the creation operator of electron at site j with spin $\sigma =\uparrow,\downarrow$ and U is the on-site repulsion. The hopping integral is engineered as $t_j = \sqrt{j(N-j)}$. It has been widely studied in connection with correlation effects in narrow-band solids and the concept of entanglement [11]. If the on-site interaction is absent, i.e., $U = 0$, it has been shown in [4, 5, 9] that an arbitrary many-particle state can be transferred to its mirror counterpart perfectly after the time $\tau = \pi/2$. It is due to the fact that the energy-level structure and the parity of the corresponding eigenstate satisfy the spectrum-symmetry matching condition (SSMC) introduced in [9]: Let ϕ_n be the common eigen-function of H and mirror operator R with the eigen-values ε_n and p_n respectively. It is easy to find that any state ψ at time τ can evolve into its symmetrical counterpart $R\psi$ if the eigenvalues ε_n and p_n match each other and satisfy the SSMC $\exp(-i\varepsilon_n\tau) = p_n$.

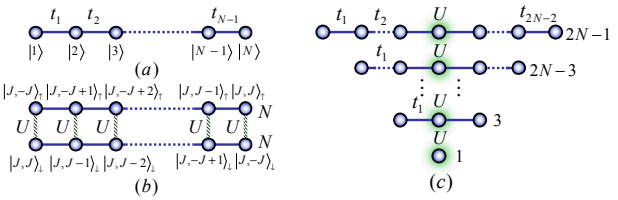

FIG. 1: *(color on line) (a) The engineered Hubbard model of N sites with two electrons. (b) The two-leg ladder of spinless Bloch electrons, which is equivalent to the above Hubbard model. The rung represents the on-site interactions. We label the sites in the two legs with a standard angular momentum basis $|JM\rangle$ in two opposite orders;(c) The above equivalent two-leg network can be further reduced according to the product representations $SO(3)\otimes SO(3)$, as the direct sum of N central potential barrier model.*

It is easy to imagine that the energy levels should be shifted by ceratin deviations from the original spectra when the on-site repulsion is switched on. Nevertheless there still exists the possibility that the new set of shifted energy levels satisfy the SSMC for an appropriate U since the certain symmetry remains as will be shown in the following discussions. We first illustrate our analysis along this direction schematically in the Suppose that there are only two electrons with opposite spin in the engineered Hubbard model (see Fig. 1a). The on-site interaction occurs only when the two electrons occupy a same site. Alternatively, the Hubbard chain is equivalent to the spinless Bloch electron network with two legs and N rungs (see Fig. 1b). The on-site interaction is denoted by the rungs. Each site in the leg corresponds to a single electron Bloch state, $|j\rangle_\uparrow = c^\dagger_{j\uparrow}|0\rangle$ or $|j\rangle_\downarrow = c^\dagger_{j\downarrow}|0\rangle$, $j = 1, 2, \cdots, N$.

According to the ref. [4], we can associate these states to the angular momentum states

$$|J, M\rangle_\uparrow = |J + M + 1\rangle_\uparrow, |J, M\rangle_\downarrow = |J - M + 1\rangle_\downarrow, \quad (2)$$

where a given $J = (N-1)/2$, $M = J, J-1, \cdots, -J+1, -J$. Then it is easy to check that for the engineered couplings, the lowing operator of angular momentum can be realized in terms of the fermion operators as

$$J^{(\uparrow)}_- = J^{(\uparrow)}_x - iJ^{(\uparrow)}_y = \sum_j t_j c^\dagger_{j,\uparrow} c_{j+1,\uparrow} \quad (3)$$

$$J^{(\downarrow)}_- = J^{(\downarrow)}_x - iJ^{(\downarrow)}_y = \sum_j t_j c^\dagger_{j+1,\downarrow} c_{j,\downarrow} \quad (4)$$

which generates the group $SO(3)$ together with $J^{(\sigma)}_+ = (J^{(\sigma)}_-)^\dagger$ and $J^{(\sigma)}_z = \sum_{j\sigma} j c^\dagger_{j,\sigma} c_{j,\sigma}$ where $\sigma = \uparrow, \downarrow$. Then we can rewrite the Hamiltonian as

$$\begin{aligned} H &= 2J^{(\uparrow)}_x + 2J^{(\downarrow)}_x + V; \\ V &= U \sum_M |J, M; J, -M\rangle \langle J, M; J, -M|, \end{aligned} \quad (5)$$

where the two-particle associated state $|J, M; J, -M\rangle$ is defined by $|J, M; J, M'\rangle = |J, M\rangle_\uparrow \otimes |J, M'\rangle_\downarrow$.

The intrinsic dynamic symmetry of the above generalized model is described as $SO(3) \otimes SO(3)$. Thus the addition theorem for two angular momenta [12] can be employed to reduce the representation of this generalized Hubbard model according to the decomposition of the product representation $D^{[J]} \otimes D^{[J]} = \sum_{L=0}^{2J} \oplus D^{[L]}$, where $D^{[J]}$ is an irreducible representation of $SO(3)$. The key point in our treatment is to expressed the on-site interaction term V as the sum of irreducible tensor operators. To this end, we use the Clebsch-Gordan coefficients $C^{LM}_{J,M_1;J,M_2} = \langle (JJ)L, JM | J, M_1; J, M_2\rangle$ to write the eigenvector of the total angular momentum $|L, M\rangle \equiv |(JJ)L, JM\rangle = \sum_{M_1+M_2=M} C^{LM}_{J,M_1;J,M_2} |J, M_1; J, M_2\rangle$. From the corresponding inverse transformation, the interaction term can be decomposed as

$$V = U \sum_{LL'} \sum_M C^{L0}_{J,M;J,-M} C^{L'0}_{J,M;J,-M} |L, 0\rangle \langle L', 0|. \quad (6)$$

Due to the orthogonal relation of Clebsch-Gordan coefficients

$$\sum_m C^{LM}_{J,m;J,M-m} C^{L'M}_{J,m;J,M-m} = \delta_{L,L'}, \quad (7)$$

the on-site interaction can be reduced as the sum of the irreducible tensors, i.e., $V = \sum_L W^{[L]} = U \sum_L |(JJ)L, 0\rangle \langle (JJ)L, 0|$.

Therefore, we have proved that the engineered Hubbard Hamiltonian can be written as the direct sum of N irreducible sub-Hamiltonians $H^{(L)} = H^{(L)}_0 + W^{[L]} = 2J_x + W^{[L]}$. The model described by each $H^{(L)}$ can be inversely mapped into a new Bloch electron model with a CPB, whose Hamiltonian is

$$H^{(L)} = \sum_{j=-L}^{L-1} (t_j a^\dagger_j a_{j+1} + h.c.) + U a^\dagger_0 a_0, \quad (8)$$

where $a^\dagger_j$ is the creation operator of new fermion and $t_j = \sqrt{(L+j+1)(L-j)}$. As illustrated in Fig. 1c, the on-site interacting qubit network is reduced into a direct sum of N Bloch electron models with CPB.

Spectrum-symmetry matching for nonzero U. To see whether the engineered Hubbard model can serves as a quantum data bus to coherently transfer quantum information, we need to study the influences of the on-site interaction on the SSMC.

Let us first recall the on-site interaction free case, where the matrix representation of the Hamiltonian (1) in single-particle subspace is equivalent to that a high spin with angular momentum $J = (N-1)/2$ precessing in transverse magnetic field. For two-electron case, the above analysis shows that the original Hamiltonian can be reduced into N single-particle engineered models on l-site chain ($l = 1, 3, ..., 2N-1$) without CPB. Obviously, all the eigenstates also meet the SSMC that guarantees a perfect QST. It is crucial for our analysis that the mirror symmetry is not broken even in presence of the on-site interaction, then the effect of nonzero U on the deviation of the energy levels determines the fidelity of QST via such system.

Now we consider the Hamiltonian (8) with nonzero U. Since $[H^{(L)}, R] = 0$, the eigenstates can be classified into two sets with different parities. When U is switched on, a set of levels reminds unchanged while other set of levels deviates by a nonzero values $\Delta_M = |E_M(U) - E_M(0)|$, (see Fig. 2). To prove this generally, we calculate the action of $W^{[L]}$ on the eigenvectors

$$\left|L, M(\frac{\pi}{2})\right\rangle = e^{i\frac{\pi}{2}J_y} |L, M\rangle = \sum_{M'} d^L_{M'M}(\frac{\pi}{2}) |L, M'\rangle \quad (9)$$

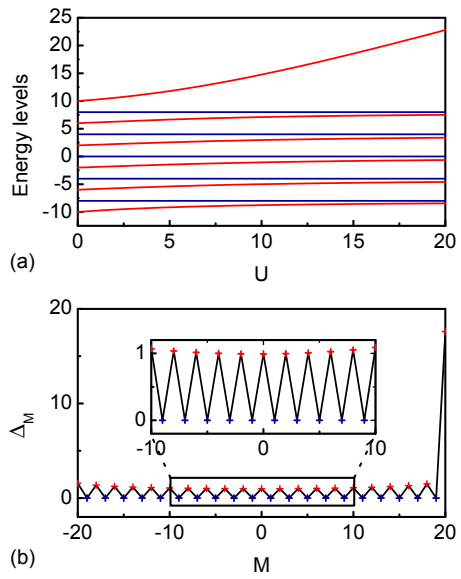

FIG. 2: *(color on line) (a) Numerical simulation of the energy levels effected by U for $H^{(L)}$ with $L = 5$, $U = 0 \sim 20$. (b) The level shift Δ_M of $H^{(L)}$ for $L = 20$ system. It shows that for small M, Δ_M is approximately uniform. Notice that for the optimal $U = 40.5$, the level shifts for small M are approximately equal to the half of the level difference with $U = 0$.*

of the reduced Hamiltonian $H_0^{(L)} = 2J_x$. Because $d^L_{M'M}(\pi/2) = (-1)^{L-M} d^L_{-M'M}(\pi/2)$ we have $d^L_{0M}(\pi/2) = 0$ for odd $L - M$, and then

$$W^{[L]} \left| L, M(\frac{\pi}{2}) \right\rangle = U d^L_{0M}(\frac{\pi}{2}) \left| L, 0 \right\rangle = 0. \qquad (10)$$

This means that states $|L, M(\pi/2)\rangle$ ($L - M$ is odd) are also the eigenstates of $H^{(L)}$. Indeed these corresponding levels are free of the on-site interaction.

Another set of eigenvalues should be shifted by the on-site interaction. Imagine that if the energy deviation Δ_M is not so sensitive to M, the shifts of the levels are approximately the same. There may exist an appropriate U to ensure that the final levels satisfy the SSMC with another greatest common divisor. It will result in the perfect QST in the invariant subspace $V^{[L]}$: $\{|L, M\rangle, M = L, L-1, \cdots, -L\}$.

In order to verify our conjecture, numerical simulation is employed for small size systems. Exact diagonalization results for $H^{(L)}$ on L lattice are plotted in Fig. 2. In Fig. 2a, the energy levels as functions of U for $L = 5$ system shows that the spectrum of $H^{(L)}$ consists of two sets of energy levels, one is independent of U, while the other is shifted by the repulsion. This conclusion is in agreement with the above analysis for arbitrary L. Numerical calculation for Δ_M of $H^{(L)}$ with $U = 40.5$ on 21-site lattice is plotted in Fig. 3b. It shows that for small M, Δ_M is approximately uniform. On the other hand, numerical calculations also indicate that in the invariant subspace $V^{[L]}$, the components of the state $|L, -L\rangle$ on the basis $V^{[L]}$ for small M are dominant, i.e, the effective levels of such state should be shifted uniformly. In other words, for such kind of initial state $|L, -L\rangle$, when U takes an appropriate value, the effective levels can satisfy the SSMC approximately. Thus state $|L, -L\rangle$ can be transferred into $|L, L\rangle$ near perfectly.

Near-perfect swap of two electrons. Now we consider the spin state swapping of two electrons located on the two ends of the one-dimensional lattice. For the initial state

$$|\psi(0)\rangle = C^\dagger_{1,\uparrow} C^\dagger_{N,\downarrow} |0\rangle = |J, -J; J, -J\rangle , \qquad (11)$$

the quantum state swapping is a mapping from $|\psi_J(0)\rangle$ to

$$|\psi(t)\rangle = C^\dagger_{N,\uparrow} C^\dagger_{1,\downarrow} |0\rangle = - |J, J; J, J\rangle \qquad (12)$$

can be approximately realized in a dynamic process since the SSMC can be satisfied for an appropriate U as discussed above.

In order to confirm the above prediction, the numerical simulation is performed for the swapping fidelity

$$F(U, t) = \left| \langle J, J; J, J| \, e^{-iHt} \, |J, -J; J, -J\rangle \right|^2 \qquad (13)$$

where H is the Hamiltonian (1) for the engineered Hubbard model. During the time range $t \in [0, 10]$, the maxima of the fidelity, $F_{\max}(U) = \max\{F(U, t), t \leqslant 10\}$ are plotted in Fig. 3a as the functions of the on-site interaction strength U for the system of $N = 4, 5$ and 6 sites. It shows that there indeed exist some U to get very high $F_{\max}(U)$, which seems like that the propagation of electrons is scattering-free. One can see for the same N that $F_{\max}$ have several regular peaks, and for each peak, the revival times T_r are different. It is interesting to find that, for the first regular peak, the revival time $T_{r1} \simeq \pi$, the second peak, $T_{r2} \simeq 1.5\pi$, ..., and the Nth peak, $T_{rn} \simeq 0.5\pi(n + 1)$. The corresponding optimized U and revival time T_r for each peak are listed in Table 1.

Peaks	U	$F_{\max}$	T_r
1	6.6	0.9847	$3.14 \simeq \pi$
2	11.6	0.9768	$4.71 \simeq 3\pi/2$
3	16.2	0.9724	$6.28 \simeq 2\pi$
4	20.6	0.9698	$7.85 \simeq 5\pi/3$
5	25.0	0.9683	$9.42 \simeq 3\pi$

Table 1

Table 1. The maxima of fidelity $F_{\max}$, the corresponding optimized U and revival time T_r obtained by numerical simulations for the engineered Hubbard model

on 4-site system.

On the other hand, for different $N = 2 \sim 10$, more detailed data, such as the optimized U, $F_{\max}$ and the corresponding revival time T_r for the first and second peaks are also obtained numerically to reveal the hidden relationship between them. In Table 2, these numerical results are listed with some obvious characters for the cases of $N = 4 \sim 10$.

	1st Peaks			2nd Peaks		
N	U	$F_{\max}$	T_r	U	$F_{\max}$	T_r
2	2.3	0.9999	2.72	3.6	0.9999	3.50
3	4.9	0.9926	3.18	8.0	0.9929	4.74
4	6.6	0.9847	3.14	11.6	0.9768	4.71
5	8.6	0.9873	3.14	15.0	0.9802	4.71
6	10.6	0.9906	3.14	18.4	0.9856	4.71
7	12.6	0.9931	3.14	21.8	0.9894	4.71
8	14.5	0.9948	3.14	25.2	0.9920	4.71
9	16.5	0.9960	3.14	28.7	0.9938	4.71
10	18.5	0.9968	3.14	32.1	0.9950	4.71

Table 2

Table 2. The maximal fidelities and the corresponding revival time T_r of the first two peaks obtained by numerical simulations for the systems with N and U.

As for the quantum swapping scheme we study here, the following prominent characters can be found from the above analysis. Firstly, the optimized U is linearly proportional to N with slopes approximately 2 and 3.4 for the first and second peak respectively. Secondly, for a given optimized U, the larger N is, the higher the maximum of the fidelity $F_{\max}$ becomes. Consequently, the QST will be better. Thirdly, as the statements above, the revival times are π and $3\pi/2$ for the first and second peak, approximately. Our further numerical results shows that the above experiential law can hold for larger N. For different peaks, the linear relations between optimized U and the sizes N are also shown in Fig. 3b.

From the Tables we set above, it is obvious that the revival times obey the experience formula $T_{rn} = 0.5\pi(n+1)$ approximately, where $n = 1, 2, ...,$ is the order of the peaks of $F_{\max}$. This formula can be understood based on the above analysis. If the effective levels are shifted by U uniformly from $\Delta_M = 0$ to 1, the possible greatest common divisors meet the SSMC are $2/(n+1)$ ($n = 1, 2, ...,$). The corresponding level shift is $2[1-1/(n+1)]$, which results in the revival period T_{rn}. A question to be asked is about the relationship between the level shifts and the repulsion U. In Fig. 3b, the optimized U as the functions of the sizes N are plotted. Interestingly, they are simply linear functions in the range we concerned.

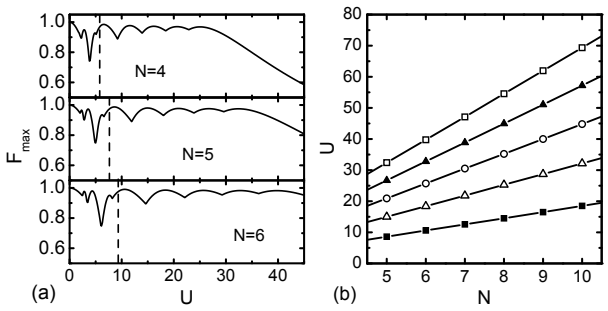

FIG. 3: *(a) Numerical simulation of the maximal $F_{\max}$ as the function of the interaction strength U for $N = 4$ sites system. (b) The optimal U as the functions of the sizes N obtained by the numerical simulations. The corresponding revival times, $T_r = \pi$ (solid square), $\pi/2$ (triangle), 2π (circle), $5\pi/2$ (solid triangle), and 3π (square). It indicates that the optimal interaction strengths U are directly proportional to N approximately.*

Summary. In summary, we study the QST in the engineered Hubbard model with on-site interaction analytically and numerically. It is proved that the system of two electrons with opposite spins in this N sites quantum network can be reduced into N one dimensional engineered single Bloch electron models with CPB rigorously. Analytic calculation and numerical results both show that the engineered Hubbard model can perform perfect quantum swapping between two distant electrons with opposite spins even when the certain Coulomb interaction exists. The angular momentum reduction method could be expected to work for the QST in an engineered quantum spin models with more electrons.

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[1] D. P. DiVincenzo, Fortsch. Phys. 48, 771 (2000) (in special issue on Experimental Proposals for Quantum Computation).
[2] S. Bose, Phys. Rev. Lett. **91**, 207901 (2003).
[3] M-H. Yung and S. Bose, Phys. Rev. A. **71**, 032310 (2005).
[4] M. Christandl, N. Datta, A. Ekert, and A. J. Landahl, Phys. Rev. Lett. **92**, 187902 (2004).
[5] C. Albanese, M. Christandl, N. Datta, and A. Ekert, Phys. Rev. Lett. **93**, 230502 (2004).
[6] T. Shi, Y. Li, Z. Song, and C.P. Sun, Phys. Rev. A **71**, 032309 (2005).
[7] Y.Li, T.Shi, B.Chen, Z.Song, C.P.Sun, Phys. Rev. A **71**,

022301 (2005).
[8] Z. Song, C. P. Sun, quant-ph/0412183, Fizika Nizkikh Temperatur, **31**, Nos. 8/9 (2005).
[9] Y.Li, Z.Song, and C.P.Sun, quant-ph/0504175.
[10] See The Hubbard Model, edited by A. Montorsi, World Scientific, Singapore, (1992).
[11] Paolo Zanardi, Phys. Rev. A **65**, 042101 (2002).
[12] L. Biedenharn and J. Louck, *The Racah-Wigner algebra in quantum theory,* Addison-Wesley, Reading MA, 1981.